# Unsteady flow through an underdrained earth dam

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An analytical treatment is given of the problem of the establishment of the flow through a dam or levee with a horizontal underdrain, when the head behind it is raised and then kept at a constant value. The essential idea employed in the analysis is to consider the unsteady flow as a time-dependent perturbation of the final steady flow. The unsteady potential $\phi(x, y, t)$ is expanded in a power series of $e^{-\lambda t}$, of the form

$$\phi(x, y, t) = \phi_0(x, y) + \phi_1(x, y)\, e^{-\lambda t} + O(e^{-2\lambda t}),$$

where $\phi_0(x, y)$ is the known steady-state potential, $\phi_1(x, y)$ is a perturbation potential and $O(e^{-2\lambda t}) = \phi_2(x, y)\, e^{-2\lambda t} + \phi_3(x, y)\, e^{-3\lambda t} + \ldots$. Each of the terms $\phi_n(x, y)\, e^{-n\lambda t}$ can be thought of as being a perturbation term of its precursor in the series, and the present approach is limited to the computation of the first perturbation term $\phi_1(x, y)\, e^{-\lambda t}$.

It is shown that $\phi_1$ satisfies Laplace's equation $\nabla^2\phi_1 = 0$ in a dimensionless hodograph plane. The free-boundary condition is linear but complicated, containing the eigenvalue $\lambda$, which is fixed by a determinantal equation. The amplitude of the displacement of the free surface is left undetermined; only the mode of the motion and the eigenvalue are computed. The results of a numerical example are summarized.

## 1. Introduction

Curle (1956) has obtained a general technique for considering the unsteady development of steady two-dimensional jet and cavity flows. The approach used was that of expanding the velocity potential $\Phi$ in ascending powers of $e^{-\lambda t}$, so that

$$\Phi(x, y, t) = \Phi_0(x, y) + \Phi_1(x, y)\, e^{-\lambda t} + O(e^{-2\lambda t}).$$

The unsteady free boundary was determined by normally displacing the steady-state boundary by an amount $\delta(s, t) = \delta_1(s)\, e^{-\lambda t} + O(e^{-2\lambda t})$. In the present paper, the author applies the above technique to a problem of free surface flow in a porous medium. Although the two flows have some essentially different features (gravity is neglected in the jet flow, kinetic energy in the seepage flow), the technique works equally well in both cases. The boundary-value problem arising in the present flow is somewhat more complicated because the trigonometric functions appearing in the free-boundary condition of the jet flow are replaced by algebraic functions, so that no recursion formula is available for the coefficients of the Fourier series solution of $\nabla^2\Phi_1 = 0$. Instead, these coefficients result from the solution of a system of linear, homogeneous equations. The reader is expected either to be familiar with the flow through porous media or not to be interested in the subject as such, since this paper only emphasizes the extension of Curle's technique and the solution of a special kind of mixed boundary-value problem.

## 2. Statement of problem and outline of method

The problem considered is that of a homogeneous dam or earth embankment, with a horizontal underdrain and behind which the level of a reservoir is gradually raised to a depth $H$ (figure 1). It is to be expected that some time will elapse before steady flow will be reached after the head $H$ has been kept at a constant value. The purpose of this paper is to analyse the flow when it approaches its final steady state.

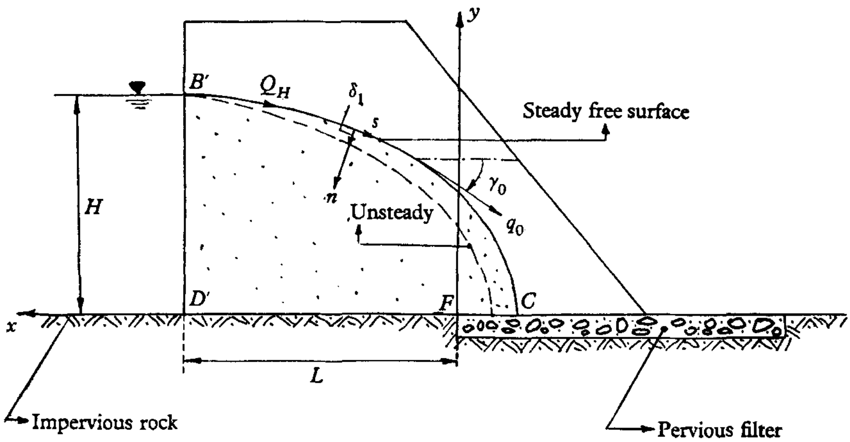


FIGURE 1. Underdrained earth dam.

It is assumed that the porous medium is completely saturated, homogeneous and isotropic, and that the flow is everywhere laminar. If the fluid is restricted to water of constant density and constant viscosity, then all the necessary conditions (Hubbert 1940) for the existence of a velocity potential $\Phi$ are satisfied:

$$\Phi = K\phi = K(y+p/\gamma), \tag{2.1}$$

$$K = k\gamma/\mu, \tag{2.2}$$

where the symbols are defined as follows: $K$, hydraulic conductivity, dimensions $[L/T]$; $k$, permeability, depending upon the medium alone, dimensions $[L^2]$; $\mu$, viscosity of water; $\gamma$, unit weight of water; $\phi$, hydraulic head; $y$, elevation head; $p/\gamma$, pressure head. $\Phi$ has the dimensions $[L^2/T]$ and can be combined with the stream function $\Psi$, $[L^2/T]$, to form the complex potential function $W = \Phi + i\Psi$. $K$ is assumed to be constant, and consequently so are $k$ and the porosity $\epsilon$ of the medium. Effects of capillarity are neglected and Darcy's law in its simplest form can then be written

$$\mathbf{q} = -K \operatorname{grad} \phi, \tag{2.3}$$

or

$$u = -\frac{\partial\Phi}{\partial x}, \quad v = -\frac{\partial\Phi}{\partial y}. \tag{2.4}$$

The insertion of **q** into the continuity equation div **q** = 0, leads to

$$\nabla^2\phi = 0. \tag{2.5}$$

Since the compressibility of the medium in unconfined flow can be neglected, (2.1), (2.3), (2.4) and (2.5) are valid as well for $\phi_0(x,y)$ as for $\phi(x,y,t)$. Steady-state values are denoted by the subscript 0, such as $q_0$; $u_0$; $v_0$; $p_0$; $\gamma_0 = \tan^{-1} v_0/u_0$; $\phi_0$; $\Phi_0$; $\Psi_0$; $W_0$. The expansion of $\phi(x,y,t)$ in a power series of $e^{-\lambda t}$, of the form

$$\phi(x,y,t) = \phi_0(x,y) + \phi_1(x,y)\,e^{-\lambda t} + O(e^{-2\lambda t}), \tag{2.6}$$

implies that $\phi_1(x,y)$ is a harmonic function, if $\phi_0(x,y)$ represents the known steady potential. In (2.3), $\lambda$ represents an eigenvalue to be determined. The perturbation potential $\phi_1(x,y)$ remains harmonic under a conformal transformation. Since the steady hodograph can be reduced to a simple configuration, it is quite natural to look for a solution of $\nabla^2\phi_1 = 0$ in such a transformed hodograph plane, provided the basic equation of motion of the unsteady free surface is transferred to the steady boundary. As in the jet flow, the unsteady boundary may be found by normally displacing the steady-state boundary by an amount

$$\delta(x,y,t) = \delta_1(x,y)\,e^{-\lambda t} + O(e^{-2\lambda t}).$$

## 3. Formulation of the boundary value problem

Consider the dimensionless hodograph plane of figure 2.

$$\chi = \alpha + i\beta = -\frac{K}{\overline{w}_0}, \tag{3.1}$$

where $\overline{w}_0 = u_0 - iv_0$; $\chi$ is simply related to $W_0$ by $W_0 = Q_H\chi$.

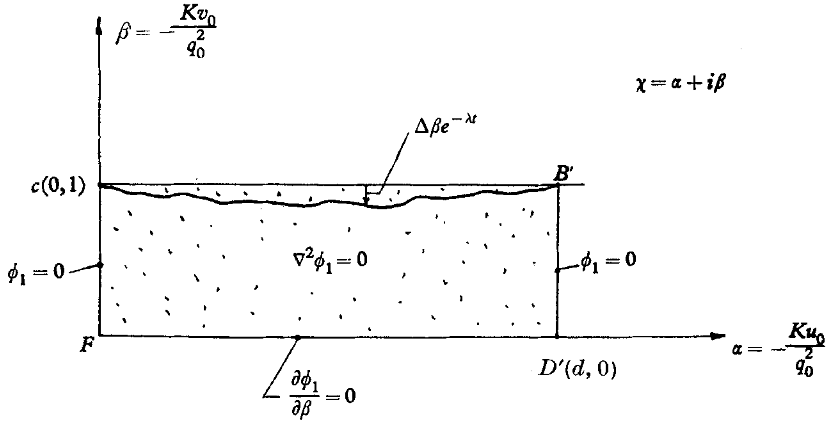


FIGURE 2. Hodograph plane $\chi = -K/\overline{w}_0$.

The abscissa $d = c(KH/Q_H)$, where $c$ is a constant depending upon the ratio $H/L$ of the dam and $Q_H$ is the seepage per unit width for a head $H$ on the dam. Application of the concept of a bounding surface (Lamb 1932, p. 7) leads to the basic equation of motion of the unsteady free surface

$$\frac{\epsilon}{K}\frac{\partial\phi}{\partial t} = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 - \frac{\partial\phi}{\partial y}. \tag{3.2}$$

Let $\phi(x, y, t)$ satisfy (3.2). Then it follows that, at the unsteady free surface,

$$-\frac{\epsilon}{K}\lambda\phi_1 e^{-\lambda t} = \phi_{0x}^2+\phi_{0y}^2-\phi_{0y}+(2\phi_{0x}\phi_{1x}+2\phi_{0y}\phi_{1y}-\phi_{1y})e^{-\lambda t}+O(e^{-2\lambda t}). \quad (3.3)$$

This condition can be transferred to one at the steady free boundary in the hodograph plane. Assume that, corresponding to the displacement $\delta_1(x, y)e^{-\lambda t}$ of the boundary in the physical plane, there is a displacement $\Delta\beta e^{-\lambda t}$ in the hodograph plane. Under this assumption, (3.3) can be transferred to the steady boundary by addition of a term $-\Delta\beta e^{-\lambda t}$. Once at the steady boundary, and only then, $\phi_{0x}^2+\phi_{0y}^2-\phi_{0y} = 0$, and, ignoring $O(e^{-2\lambda t})$, (3.3) reduces to

$$-\frac{\epsilon}{K}\lambda\phi_1 = 2\phi_{0x}\phi_{1x}+2\phi_{0y}\phi_{1y}-\phi_{1y}-\Delta\beta. \quad (3.4)$$

Now,

$$q_0\phi_{1s} = -K(\phi_{1x}\phi_{0x}+\phi_{1y}\phi_{0y}). \quad (3.5)$$

Recall that $W_0 = \Phi_0+i\Psi_0 = Q_H\alpha+Q_H\beta i$, so that along the steady free surface

$$\left.\begin{aligned}\frac{\partial}{\partial s} &= -q_0\frac{\partial}{\partial\Phi_0} = -\frac{q_0}{Q_H}\frac{\partial}{\partial\alpha},\\ \frac{\partial}{\partial n} &= -q_0\frac{\partial}{\partial\Psi_0} = -\frac{q_0}{Q_H}\frac{\partial}{\partial\beta},\\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial\Phi_0}\frac{\partial\Phi_0}{\partial y} = -\frac{v_0}{Q_H}\frac{\partial}{\partial\alpha}.\end{aligned}\right\} \quad (3.6)$$

Then (3.4) reduces via (3.5) and (3.6) to

$$-\frac{\epsilon}{K}\lambda\phi_1 = \frac{2}{K}\frac{q_0^2}{Q_H}\frac{\partial\phi_1}{\partial\alpha}+\frac{v_0}{Q_H}\frac{\partial\phi_1}{\partial\alpha}-\Delta\beta. \quad (3.7)$$

With

$$\Delta\beta = \frac{\partial\beta}{\partial n}\delta_1 = -\frac{q_0}{Q_H}\delta_1 \quad\text{and}\quad q_0^2 = -Kv_0,$$

(3.7) reduces to

$$\frac{\epsilon}{K}\lambda\phi_1-\frac{v_0}{Q_H}\frac{\partial\phi_1}{\partial\alpha}+\frac{q_0}{Q_H}\delta_1 = 0. \quad (3.8)$$

A second relation between $\delta_1$ and $\phi_1$ can be found to hold at the free surface by mathematically expressing that this surface is a material line. The unsteady free surface is not a streamline but has a velocity component normal to the boundary. This component is due:

(*a*) To changes in $\delta_1(x, y)e^{-\lambda t}$ because of the exponential decrement with time. These changes, at any $(x, y)$ are expressed as

$$\delta_1(x, y)e^{-\lambda(t+\Delta t)}-\delta_1(x, y)e^{-\lambda t} \approx -\lambda\Delta t\delta_1 e^{-\lambda t}.$$

The corresponding displacement of the boundary, of order $e^{-\lambda t}$, is equal to

$$-\frac{K}{\epsilon}\frac{\partial\phi_1}{\partial n}e^{-\lambda t}\Delta t.$$

It follows that

$$\frac{K}{\epsilon}\frac{\partial\phi_1}{\partial n} = \lambda\delta_1. \quad (3.9)$$

(*b*) To changes in $\delta_1 e^{-\lambda t}$ because $s$ changes as the fluid particles are convected along the free surface. They can be expressed as

$$\left[\delta_1\left(s+\frac{q_0}{\epsilon}\Delta t\right)-\delta_1(s)\right]e^{-\lambda t}\approx\frac{q_0}{\epsilon}\frac{d\delta_1}{ds}\Delta t\, e^{-\lambda t},$$

so that

$$K\frac{\partial\phi_1}{\partial n}=-q_0\frac{d\delta_1}{ds}. \tag{3.10}$$

It is to be expected that in the case of a slow and gradual rise of the level behind the dam, the component given by (3.10) is small compared to the component of (3.9). The boundary-value problem will be set up and solved under the assumption that component (3.10) can be neglected.

Application of (3.6) to (3.9) leads to

$$\frac{K}{\epsilon}\frac{q_0}{Q_H}\frac{\partial\phi_1}{\partial\beta}=-\lambda\delta_1. \tag{3.11}$$

Elimination of $\delta_1$ from (3.8) and (3.11) gives

$$\frac{\partial\phi_1}{\partial\beta}=\frac{\epsilon^2}{K^2}\frac{Q_H^2}{q_0^2}\lambda^2\phi_1-\frac{\epsilon}{K}\frac{Q_H}{q_0^2}\lambda v_0\frac{\partial\phi_1}{\partial\alpha}. \tag{3.12}$$

Note that

$$\alpha=-\frac{Ku_0}{u_0^2+v_0^2},\quad \beta=-\frac{Kv_0}{u_0^2+v_0^2},\quad q_0^2=\frac{K^2}{\alpha^2+\beta^2}, \tag{3.13}$$

and introduce $a=\epsilon Q_H/K^2$, dimension $[T]$, to find at the free surface ($\beta=1$):

$$\frac{\partial\phi_1}{\partial\beta}=(a\lambda)^2(1+\alpha^2)\phi_1+(a\lambda)\frac{\partial\phi_1}{\partial\alpha}. \tag{3.14}$$

With the free-boundary condition (3.14), the problem is mathematically formulated. It is convenient to consider $a\lambda$ as a dimensionless parameter.

## 4. The solution

If there is a solution $\phi_1$, it must satisfy the condition

$$\int_0^d(-\phi_{1\beta})_{\beta=1}\,d\alpha+\int_0^1(-\phi_{1\alpha})_{\alpha=0}\,d\beta+\int_0^1(\phi_{1\alpha})_{\alpha=d}\,d\beta=0, \tag{4.1}$$

expressing that the perturbation flow must satisfy the continuity requirement that the mass flux of the perturbation flow entering the fluid-filled region through the free surface equals that leaving it through the entrance surface and the drain. This perturbation flow vanishes like $e^{-\lambda t}$ and is superimposed on the steady flow. Separation of variables for $\nabla^2\phi_1=0$ in the strip of figure 2 leads to a solution of the form

$$\phi_1=\sum_{n=1}^{\infty}A_n\cosh\frac{n\pi}{d}\beta\sin\frac{n\pi}{d}\alpha. \tag{4.2}$$

This solution satisfies the boundary conditions along $FD'$, $D'B'$ and $FC$; $A_n$ is left to be determined by the boundary condition along $CB'$. It follows from (4.1) and some integrations that the dimensionless parameter $a\lambda$ now has to satisfy the condition

$$(a\lambda)^2 = \frac{-\sum_1^\infty A_n \sinh\frac{n\pi}{d} + \sum_1^\infty (-1)^n A_n \sinh\frac{n\pi}{d}}{-\sum_1^\infty A_n \frac{d}{n\pi}[1-(-1)^n]\cosh\frac{n\pi}{d} + \sum_1^\infty A_n \left(\frac{d}{n\pi}\right)^3 [(n^2\pi^2-2)(-1)^n+2]\cosh\frac{n\pi}{d}}. \tag{4.3}$$

Although from (4.3), the existence of a solution cannot be guaranteed, the value $a\lambda = 0$ is excluded by consideration of the properties of eigenvalues. In fact, from the nature of the eigenvalue problem and the physics of the flow, we may expect an infinite number of eigenvalues to satisfy (4.3). This result would be analogous to the infinite number of eigenvalues found by Curle in the unsteady jet problem. The smallest of these eigenvalues, $\lambda_{\min}$, will determine the slowest rate of decay. In itself, (4.3) does not determine $a\lambda$, since the $A_n$ are unknown as yet and, moreover, will contain $a\lambda$. It constitutes a check on the computation of the $A_n$ and this has been verified in a numerical example.

The boundary condition of (3.14) can be written as

$$\sum_1^\infty A_n \frac{n\pi}{d}\sinh\frac{n\pi}{d}\sin\frac{n\pi}{d}\alpha = (a\lambda)^2(1+\alpha^2)\sum_1^\infty A_n\cosh\frac{n\pi}{d}\sin\frac{n\pi}{d}\alpha + (a\lambda)\sum_1^\infty A_n\frac{n\pi}{d}\cosh\frac{n\pi}{d}\cos\frac{n\pi}{d}\alpha. \tag{4.4}$$

Multiply each side by $\sin m\pi\alpha/d$ and integrate over the interval $(0, d)$, to obtain

$$\frac{d}{2}A_m\frac{m\pi}{d}\sinh\frac{m\pi}{d} = (a\lambda)^2\sum_1^\infty\left[\int_0^d (1+\alpha^2)\sin\frac{m\pi\alpha}{d}\sin\frac{n\pi\alpha}{d}\,d\alpha\right]A_n\cosh\frac{n\pi}{d} + (a\lambda)\sum_1^\infty\left[\int_0^d \cos\frac{n\pi\alpha}{d}\sin\frac{m\pi\alpha}{d}\,d\alpha\right]A_n\frac{n\pi}{d}\cosh\frac{n\pi}{d}. \tag{4.5}$$

Let

$$b_{m,n} = \int_0^d (1+\alpha^2)\sin\frac{m\pi\alpha}{d}\sin\frac{n\pi\alpha}{d}\,d\alpha, \quad c_{m,n} = \int_0^d \cos\frac{n\pi\alpha}{d}\sin\frac{m\pi\alpha}{d}\,d\alpha. \tag{4.6}$$

Then, (4.5) can be written

$$\frac{d}{2}A_m\frac{m\pi}{d}\sinh\frac{m\pi}{d} = (a\lambda)^2\sum_1^\infty A_n b_{m,n}\cosh\frac{n\pi}{d} + (a\lambda)\sum_1^\infty A_n\frac{n\pi}{d}c_{m,n}\cosh\frac{n\pi}{d}. \tag{4.7}$$

Introduce a new coefficient

$$a_m = \frac{A_m}{2}m\pi\sinh\frac{m\pi}{d}, \tag{4.8}$$

and write (4.7) as

$$a_m = \sum_{n=1}^\infty a_n h_{m,n}(a\lambda) \quad (m = 1, 2, \ldots), \tag{4.9}$$

where
$$h_{m,n}(a\lambda) = (a\lambda)^2\frac{2}{n\pi}b_{m,n}\coth\frac{n\pi}{d} + (a\lambda)\frac{2}{d}c_{m,n}\coth\frac{n\pi}{d}. \tag{4.10}$$

The integrations of (4.6) lead to

$$\left.\begin{aligned} b_{m,n} &= \frac{d^3}{\pi^2(m-n)^2}\cos\pi(m-n) - \frac{d^3}{\pi^2(m+n)^2}\cos\pi(m+n) \quad (m \neq n),\\ c_{m,n} &= -\frac{d}{2(m-n)\pi}[\cos\pi(m-n)-1] - \frac{d}{2(m+n)\pi}[\cos\pi(m+n)-1] \quad (m \neq n),\\ b_{m,m} &= \frac{d}{2}+\frac{d^3}{6}-\frac{d^3}{4m^2\pi^2}, \quad c_{m,m} = 0. \end{aligned}\right\} \tag{4.11}$$

The equations (4.9) represent a system of linear homogeneous equations, infinite in number and with an infinite number of unknowns $a_1, a_2, a_3 \dots$. This system can be written explicitly as

$$\left.\begin{aligned} [h_{11}(a\lambda)-1]\,a_1 + h_{12}(a\lambda)\,a_2 + h_{13}(a\lambda)\,a_3 + \dots &= 0,\\ h_{21}(a\lambda)\,a_1 + [h_{22}(a\lambda)-1]\,a_2 + h_{23}(a\lambda)\,a_3 + \dots &= 0,\\ h_{31}(a\lambda)\,a_1 + h_{32}(a\lambda)\,a_2 + [h_{33}(a\lambda)-1]\,a_3 + \dots &= 0,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots& \end{aligned}\right\} \tag{4.12}$$

In order to have a nontrivial solution of this system of equations, its determinant should vanish. Thus the concept of the infinite determinant as developed by Hill in his Lunar Theory appears in this analysis. Conditions for the convergence of such a determinant are given by Whittaker & Watson (1947, pp. 36, 413–17). The equations (4.12) are similar to Hill's equations except for the range of $n$ in $a_n$. In Hill's case, $n$ assumes the values $n = \dots -2, -1, 0, 1, 2, \dots$, while the parameter $a\lambda$ corresponds to Hill's $\mu$. It was shown by Hill that, for the purposes of his astronomical problem, a remarkably good approximation to the value of $\mu$ could be obtained by considering only the three central rows and columns of his determinant. In a numerical example $a\lambda$ has been determined within 2 % by considering only the smallest positive root of the equations $D_4(a\lambda) = 0$, $D_6(a\lambda) = 0$, $D_8(a\lambda) = 0$, where the subscripts 4, 6, 8 denote the degree of the equations in $a\lambda$ which result from the successive consideration of the determinants two by two (rows and columns), three by three, four by four, of the matrix

$$\begin{pmatrix} h_{11}-1 & h_{12} & h_{13} & h_{14} & \dots\\ h_{21} & h_{22}-1 & h_{23} & h_{24} & \dots\\ h_{31} & h_{32} & h_{33}-1 & h_{34} & \dots\\ h_{41} & h_{42} & h_{43} & h_{44}-1 & \dots\\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{4.13}$$

A very good approximation to the value of $a\lambda$ is even determined by $D_2(a\lambda) = h_{11}(a\lambda) - 1 = 0$. Once the parameter $a\lambda$ is determined, one can compute the coefficients $h_{m,n}$ from (4.10) and (4.11) and solve the system (4.12) for $a_1, a_2, a_3, \dots$. Next the coefficients $A_n$ of (4.2) are computed by (4.8) and with this the problem is solved. In particular, the mode of motion of the unsteady free surface can be derived from the equation (3.11), viz.

$$\delta_1 = -\frac{1}{\lambda}\frac{K}{\epsilon}\frac{q_0}{Q_H}\frac{\partial\phi_1}{\partial\beta} = \frac{1}{(\lambda a)}\sin\gamma_0\frac{\partial\phi_1}{\partial\beta}, \tag{4.14}$$

since $q_0 = -K \sin \gamma_0$. In case the component (3.10) cannot be neglected, as in the case of a sudden rise of the water level in the reservoir, then

$$\frac{K}{\epsilon}\frac{\partial \phi_1}{\partial n} = -\frac{q_0}{\epsilon}\frac{d\delta_1}{ds} + \lambda\delta_1, \tag{4.15}$$

or, by the use of (3.6),

$$\frac{K}{\epsilon}\left(-\frac{q_0}{Q_H}\frac{\partial \phi_1}{\partial \beta}\right) = \frac{q_0^2}{\epsilon Q_H}\frac{\partial \delta_1}{\partial \alpha} + \lambda\delta_1. \tag{4.16}$$

Insert the value of $\delta_1$ from (3.8) into (4.16) and use (3.13) to find

$$(\phi_{1\beta})_{\beta=1} = \left[(a\lambda)\frac{\alpha}{1+\alpha^2} + (a\lambda)^2(1+\alpha^2)\right]\phi_1 + \left[2a\lambda - \frac{\alpha}{(1+\alpha^2)^2}\right]\phi_{1\alpha} + \frac{\phi_{1\alpha\alpha}}{1+\alpha^2}. \tag{4.17}$$

The study of the problem with this free-boundary condition can be done exactly in the same way as before although the computations of $h_{m,n}(a\lambda)$ become tedious.

## 5. A numerical example

The mode of motion of the free surface and the exponential law have been computed for the family of dams for which $d = 3$ (De Wiest 1959). These computations can be simply repeated for any other numerical value of $d$ corresponding to the particular dam that must be investigated. The smallest positive root of $D_2(a\lambda) = 0$ and $D_4(a\lambda) = 0$ has been computed algebraically, whereas the smallest positive root of $D_6(a\lambda) = 0$ and $D_8(a\lambda) = 0$ has been determined graphically on a large-scale plot, a reduced copy of which is given in figure 3. The smallest positive roots of the successive determinants are

| Determinant | $D_2(a\lambda)$ | $D_4(a\lambda)$ | $D_6(a\lambda)$ | $D_8(a\lambda)$ | |
|---|---|---|---|---|---|
| $a\lambda$ | 0·479 | 0·492 | 0·480 | 0·490 | (5.1) |

From this one may assume that the smallest root of $D_{2n}(a\lambda) = 0$ will converge to the smallest root of $\Delta(a\lambda) = 0$, as $n \to \infty$. The values of $a_1, a_2, a_3, a_4, \ldots$ computed from the system (4.12) will differ if one considers two, three, four, ..., equations and the corresponding value of $a\lambda$ which makes their determinant vanish, because these values of $a\lambda$ vary slightly. However, it is found that the values of the $a_n$ decrease very rapidly, so that it is sufficient to consider only the coefficients $A_1$ and $A_2$ in (4.2).

A sketch of the mode of motion of the free surface is given in figure 4, for $d = 3$ and $a\lambda = 0{\cdot}49$. It can be verified that the free surface is no longer a parabola as in the steady state and that component (3.10) is small compared to component (3.9). Also, the value of $a\lambda = 0{\cdot}49$ satisfies the continuity condition (4.3). In figure 4, $c^*$ denotes the undetermined amplitude of the displacement of the free surface.

It is felt that further theoretical investigations or experiments may lead to the relation between the unknown amplitude and the speed at which the water level behind the dam rises.

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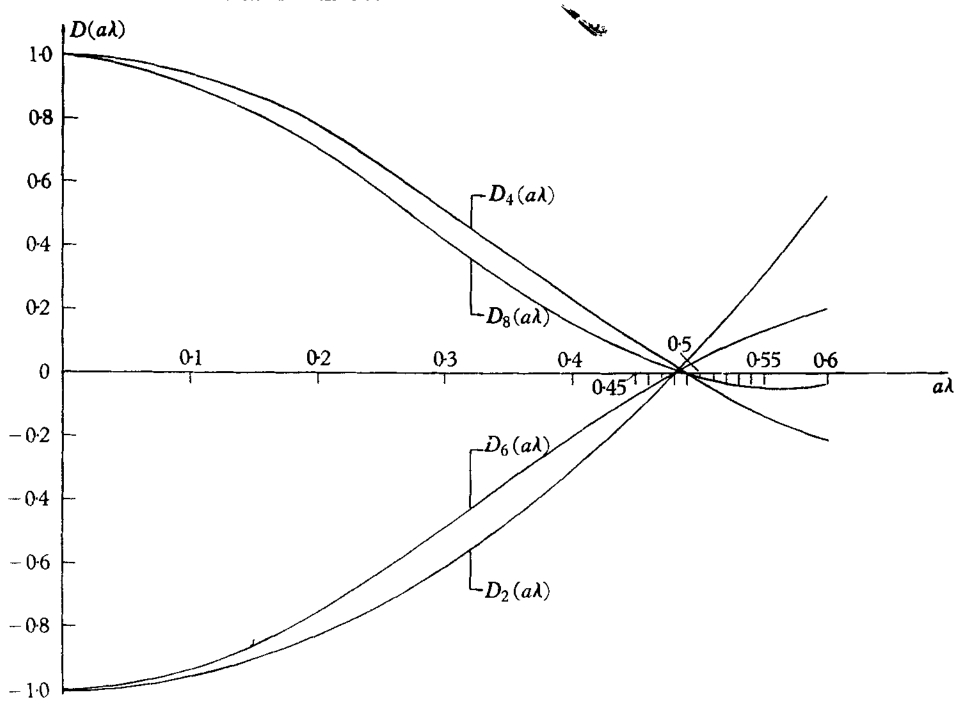


FIGURE 3. Determination of $a\lambda$.

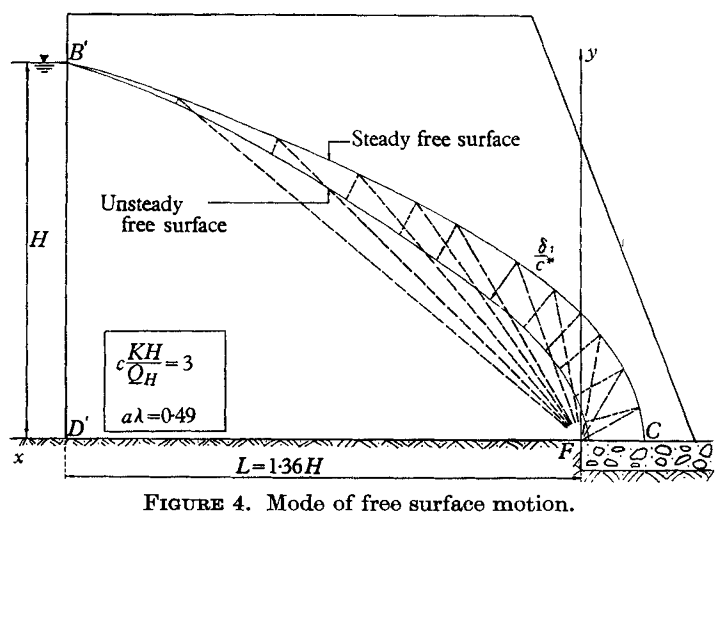


FIGURE 4. Mode of free surface motion.